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1988 J. Phys. A: Math. Gen. 21 4305

(http://iopscience.iop.org/0305-4470/21/23/014)

J. Phys. A: Math. Gen. **21** (1988) 4305–4321. Printed in the UK

# Scattering operators on Fock space: IV. The algebra of operators commuting with an internal symmetry

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Received 18 January 1988, in final form 19 July 1988

**Abstract.** An algebra of operators commuting with a given compact internal symmetry group action on a Fock space is introduced as a means for constructing unitary invariant operators on the Fock space. The algebra is generated by invariant polynomials and, in general, is an infinite-dimensional Lie algebra with a Cartan–Weyl structure. As an example, the algebra $A_2^{SO(3)}$ generated by the $l=2$ representation of SO(3) is analysed. Irreducible representations of $A_2^{SO(3)}$ are given by raising operators acting on lowest-weight states; the coefficients which connect states generated by raising operators are computed. Some of the multiplicity appearing in an irreducible representation is dealt with by introducing an $SL(2,\mathbb{R})$ subalgebra of $A_2^{SO(3)}$.

## 1. Introduction

This paper continues the investigation of unitary invariant operators on a symmetric Fock space generated by a compact internal symmetry group begun in Klink (1985, 1987a, b) (hereafter referred to as I, II and III, respectively). We call such unitary invariant operators scattering operators, in analogy with operators that are unitary and invariant on a Fock space generated by unitary representations of the Poincaré group.

A representation of scattering operators can be given by introducing the algebra of operators A that commutes with the compact group action on the Fock space. A scattering operator $S$ can then be written as $e^{i\hat{\eta}}$, where $\hat{\eta}$ is the phase operator. If $\hat{\eta}$ is Hermitian and a polynomial in elements of A, then $S$ will automatically be unitary and invariant on the Fock space. The main goal of this paper is to investigate the algebra of operators A that commutes with the underlying symmetry group.

There are a number of reasons for investigating such algebras of operators. Moshinsky (1984) and Moshinsky and Quesne (1970, 1971) have introduced the notion of complementary groups, and more recently Howe (1985) has classified such groups (he calls them dual pairs). Given a Fock space generated by the fundamental representation of $SO(n)$ (they also investigate $SU(n)$ and $Sp(n)$), they find that such a dual Lie algebra of operators commuting with the $SO(n)$ action on the Fock space is the Lie algebra of $SL(2,\mathbb{R})$. In this paper we will generalise their results by asking: given a Fock space generated by any (finite-dimensional) representation of a compact group, what is a dual algebra of operators A commuting with the compact group action? It will turn out that, in general, A no longer is a finite-dimensional Lie algebra; nevertheless, A still has a Cartan–Weyl structure, with diagonal and raising and lowering operators. Section 2 is devoted to a general analysis of A. Use is made of the isomorphism between symmetric Fock spaces and holomorphic Hilbert spaces introduced in II, so that all operators become multiplication and differentiation operators acting on polynomials of complex variables.

Aside from its mathematical interest, the main reason for generalising the notion of complementary or dual pairs of groups is its relationship with the construction of scattering operations in particle physics. We are interested in constructing unitary invariant scattering operators on a Fock space generated by representations of the Poincaré group and an internal symmetry group. The algebra of operators that commutes with these groups should provide a convenient starting point for a phenomenological analysis of, for example, pion-nucleon scattering and production data.

A third purpose of this paper is to generalise methods introduced by Arima and Iachello (1978) and Iachello and Levine (1982), in which the Hamiltonian of a non-relativistic quantum system such as a nuclear Hamiltonian or the Hamiltonian for a triatomic molecule is written in terms of boson operators on a symmetric Fock space. The quantum system specifies an $N$-dimensional multiplet of a given compact symmetry group (the s and d bosons in the interacting boson model), and then chains of subgroups between $SU(N)$ and the underlying symmetry group are introduced to deal with multiplicity. By using the algebra of operators A that commutes with the symmetry group, no chain of subgroups is required and all multiplicity is dealt with via the representations of A. This is shown in an example in § 3, where the infinite-dimensional Lie algebra of operators $A_2^{SO(3)}$ is constructed on the Fock space generated by the $l=2$ representation of SO(3). In § 4 irreducible representations of $A_2^{SO(3)}$ are constructed from raising operators acting on lowest-weight states.

## 2. Holomorphic Hilbert spaces and generalised duality

Let K be a compact internal symmetry group with a (not necessarily irreducible) representation space $V$ of dimension $N$. The many-particle symmetric Fock space $\mathcal{S}(V)$ is defined as

$$\mathcal{S}(V)=\sum_{n=0}^{\infty}\oplus(V\otimes\ldots\otimes V)^n_{\text{sym}}$$

where $(V\otimes\ldots\otimes V)^n_{\text{sym}}$ is the $n$-fold symmetrised tensor product of $V$.

We want to find the dual Lie algebra $A_V^K$ of operators that commutes with the action of K on $\mathcal{S}(V)$; this algebra of operators is much larger than just the Casimir operators of K. To find $A_V^K$ it is convenient to introduce a holomorphic Hilbert space $HL_N^2$ (Cook 1953, Segal 1956) which is isomorphic with $\mathcal{S}(V)$. Here $HL_N^2$ is the space of holomorphic functions in $N$ complex variables, $z=(z_1,\ldots,z_N)$, with norm

$$\|f\|^2=\frac{1}{\pi^N}\int_{\mathscr{C}_N}\mathrm{d}z_1,\ldots,\mathrm{d}z_N\exp\left(-\sum_{j=1}^{N}|z_j|^2\right)|f(z)|^2<\infty. \tag{1}$$

The normalisation in equation (1) agrees with that given by Bargmann (1962). This choice is motivated by the requirements given in II, equation (3), namely

$$\int \mathrm{d}\mu_G\equiv\frac{1}{\pi^N}\int_{\mathscr{C}_N}\mathrm{d}z_1,\ldots,\mathrm{d}z_N\exp\left(-\sum_{j=1}^{N}|z_j|^2\right)=1$$

$$\int \mathrm{d}\mu_G\exp\left(-\sum_{j=1}^{N}\bar z_j w_j\right)=\exp\left(-\sum_{j=1}^{N}|w_j|^2/4\right).$$

This normalisation also agrees with the differentiation inner product given in Klink and Ton-That (1979).

If $\hat{e}_1, \ldots, \hat{e}_N$ is an orthonormal basis for $V$, an orthogonal basis in the $n$-particle subspace of $\mathscr{S}(V)$ is $\hat{e}_{i_1} \otimes \ldots \otimes \hat{e}_{i_n}|_{\text{sym}}$. The correspondence between $\mathscr{S}(V)$ and $HL^2_N$ on each $n$-particle subspace is then given by

$$\hat{e}_{i_1} \otimes \ldots \otimes \hat{e}_{i_n}|_{\text{sym}} \rightarrow z_{i_1} \ldots z_{i_n}.$$

Such $n$-particle orthogonal basis elements can also be written as

$$z_1^{n_1} \ldots z_N^{n_N} \qquad \sum_{j=1}^{N} n_j = n.$$

Creation and annihilation operators take on a very simple form in $HL^2_N$, namely

$$a_i^\dagger \rightarrow z_i$$

$$a_i \rightarrow \frac{\partial}{\partial z_i} = D_i$$

with $[a_i, a_j^\dagger] = \delta_{ij}$.

The action of an element $g$ in $\mathrm{GL}(N, \mathscr{C})$ on $f$ in $HL^2_N$ is given by

$$(\Gamma_g f)(z) = f(g^{-1}z) \tag{2}$$

and the infinitesimal action is

$$(\Gamma_{ij} f)(z) = (z_i D_j f)(z). \tag{3}$$

Each $n$-particle subspace of $HL^2_N$ is an irreducible subspace of $\mathrm{GL}(N, \mathscr{C})$ denoted by $(n, 0, \ldots, 0)$ in the Gel'fand notation.

Since the compact Lie groups of interest in this paper are subgroups of $\mathrm{GL}(N, \mathscr{C})$, their action on $HL^2_N$ can be written as

$$(\Gamma_k f)(z) = f(D^{-1}(k)z). \tag{4}$$

Here $D(k)$ is a matrix element of $k \in \mathrm{K}$ with respect to the basis $\hat{e}_1, \ldots, \hat{e}_N$ in $V$.

To find the algebra $\mathrm{A}_V^{\mathrm{K}}$ of operators commuting with $\Gamma_k$ on $HL^2_N$, we first look for functions in $HL^2_N$ that are invariant with respect to K, i.e. $\Gamma_k f = f \; \forall k \in \mathrm{K}$. Since polynomials form a basis in $HL^2_N$, we look for a minimum set of invariant polynomials that generate all of the invariant polynomials. That such a set is finite is guaranteed by a theorem of Weyl (1946); see also Fogarty (1969). The easiest way to find such polynomials is to look for the identity representation of K in the $m$-fold tensor product subspaces. If the identity of K occurs in the $m$-fold tensor product, an invariant polynomial can be written as

$$p^{(m)}(z) = \sum_{i_1 \ldots i_m} C_0^0 \, {}^{V \ldots V}_{i_1 \ldots i_m} z_{i_1} \ldots z_{i_m} \tag{5}$$

the superscript on $p$ denoting the degree of the polynomial. $C_0^0 \, {}^{V \ldots V}_{i_1 \ldots i_m}$ is a Clebsch–Gordan coefficient and is discussed in more detail in appendix 1.

Assume now that a finite set of polynomials $\{p^{(m)}\}$ has been found that are invariant under $K$ and generate all of the invariant polynomials in $HL^2_N$. Define raising operators

$$(X^{+m} f)(z) \equiv p^{(m)}(z) f(z) \tag{6}$$

and lowering operators

$$(X^{-m}f)(z) \equiv p^{(m)}(D)f(z) \tag{7}$$

where $p^{(m)}(D)$ means replacing each entry $z_1, \ldots, z_N$ in $p^{(m)}(z)$ with its corresponding differential operator, $D_1 = \partial/\partial z_1, \ldots, D_N = \partial/\partial z_N$. It is easily checked that $X^{-m}$ is the adjoint of $X^{+m}$ with respect to the inner product, equation (1).

Both the raising operators $X^{+m}$ and lowering operators $X^{-m}$ are elements of $\mathrm{A}_V^{\mathrm{K}}$. New elements of $\mathrm{A}_V^{\mathrm{K}}$ are obtained from the commutators $[X^{+m}, X^{-m'}]$. (Note that $[X^{+m}, X^{+m'}] = 0$.) By continuing to commute the new elements thus generated with the original raising and lowering operators, one arrives at either a finite-dimensional Lie algebra or the commutators do not close, resulting in an infinite-dimensional Lie algebra.

The irreducible representations of $\mathrm{A}_V^{\mathrm{K}}$ can be used to distinguish the equivalent representations of K on $HL_N^2$. That is, the representation $\Gamma_k$, equation (4), is highly reducible; since the operators in $\mathrm{A}_V^{\mathrm{K}}$ by construction commute with $\Gamma_k$, they will transform among the equivalent irreducible representations of K.

Let $p_{|\chi i n_{\min}\eta\rangle}(z) \in HL_N^2$ be the polynomial transforming as the irreducible representation $\chi$ of K, with $i$ the label for the eigenvalues of a complete set of commuting observables in $\chi$. $n_{\min}$ is the smallest particle subspace in which $\chi$ occurs and $\eta$ is a multiplicity label needed if $\chi$ occurs more than once in the $n_{\min}$-particle subspace. Then a tower of polynomials all transforming as $(\chi, i)$ are generated by writing

$$\prod_m (X^{+m})^{k_m} p_{|\chi i n_{\min}\eta\rangle}(z) \qquad k_m = \text{non-negative integer.}$$

If $\mathrm{A}_V^{\mathrm{K}}$ is the largest algebra of operators commuting with $\Gamma_k$, then $\eta$ should be given by the eigenvalues of operators that commute among themselves in $\mathrm{A}_V^{\mathrm{K}}$. Such operators are generated from commutators of the form $[X^{-m}, X^{+m}]$. Included in these operators is the number operator

$$\hat{n} = \sum_{j=1}^{N} z_j D_j \tag{8}$$

whose eigenvalues are the non-negative integers labelling the particle subspaces. In general, there will be other operators in the commutators of $[X^{-m}, X^{+m}]$ that we denote by $X_\alpha^0$; the superscript 0 indicates that $X_\alpha^0$ does not change the total particle number, as do the $X^{\pm m}$. $\alpha$ distinguishes the different $X^0$ operators. Then

$$[X_\alpha^0, X_\beta^0] = 0 \tag{9}$$

since, if to the contrary, the commutators were of the form

$$[X_\alpha^0, X_\beta^0] = c_{\alpha\beta}\hat{n} + \sum_\gamma d_{\alpha\beta}^\gamma X_\gamma^0 + \sum_m e_{\alpha\beta}^m X^{+m} X^{-m}$$

upon taking the adjoint of this equation, the left-hand side would go to its negative value, while the right-hand side would not change, which is only possible if $[X_\alpha^0, X_\beta^0] = 0$ and $[\hat{n}, X_\alpha^0] = 0$.

The algebra $\mathrm{A}_V^{\mathrm{K}}$ of operators commuting with K thus has a generating set consisting of raising and lowering operators $X^{\pm m}$, and operators $X_\alpha^0$, $\hat{n}$ which commute among themselves. These operators generate a Cartan subalgebra, whose eigenvalues can be used to label elements in the irreducible representation spaces of $\mathrm{A}_V^{\mathrm{K}}$. An example of an infinite-dimensional algebra which has this structure is given in § 3.

As an example of a finite-dimensional Lie algebra, we consider the group K chosen in II, namely SO($N$), with action

$$(\Gamma_R f)(z) = f(R^{-1}z) \qquad R \in \mathrm{SO}(N), f \in HL_N^2. \tag{10}$$

Here the underlying representation space $V$ is just the fundamental $N$-dimensional representation space of SO($N$). If the symmetric tensor product of $V$ with itself is taken, the identity representation occurs once. The polynomial $p^{(2)}(z) = \Sigma_{i=1}^N z_i^2$ is clearly invariant with respect to SO($N$) and of degree 2; all other invariant polynomials are powers of $p^{(2)}(z)$. A raising and lowering operator can then be defined as

$$\begin{aligned}(X^{+2}f)(z) &= p^{(2)}(z)f(z)\\ (X^{-2}f)(z) &= p^{(2)}(D)f(z) = \sum_{i=1}^N \frac{\partial^2}{\partial z_i^2} f(z).\end{aligned} \tag{11}$$

A new element of $\mathrm{A}_N^{\mathrm{SO}(N)}$ is obtained from the commutator

$$[X^{-2}, X^{+2}] = 2N + 4\hat{n} \tag{12}$$

where $\hat{n}$ is the number operator, equation (8). The elements $\{X^{\pm 2}, \hat{n}\}$ form a basis for the Lie algebra of SL(2, $\mathbb{R}$).

For $N > 2$ each representation $(l, 0, \ldots, 0)$ of SO($N$) occurs with infinite multiplicity in $HL_N^2$; as discussed in II, these equivalent representations of SO($N$) are distinguished by $n$, the eigenvalue of the number operator $\hat{n}$, with the smallest value of $n$, $n_{\min}$, given by $n_{\min} = l$, and $n = l, l+2, l+4, \ldots$. These infinite-dimensional representations of $\mathrm{A}_N^{\mathrm{SO}(N)}$ are the discrete series of representations of SL(2, $\mathbb{R}$).

For $N = 2$, the group K = SO(2) has a natural two-dimensional reducible representation, with 'charges' $Q = \pm 1$ as basis labels in $V$, as discussed in I, § 4. Here we note that it is more convenient to choose for $z_1$ and $z_2$ of $HL_2^2$ the labels $z_+$ and $z_-$; then the polynomial $p^{(2)}(z) = z_+z_-$ is invariant under the SO(2) action defined by

$$(\Gamma_\theta f)(z) = f(\mathrm{e}^{\mathrm{i}\theta}z_+, \mathrm{e}^{-\mathrm{i}\theta}z_-).$$

The algebras $\mathrm{A}_N^{\mathrm{SO}(N)}$ just discussed are finite-dimensional Lie algebras of the sort discussed by Howe (1985) and Moshinsky and Quesne (1970, 1971). The main point of this paper is to show that a much larger class of algebras, which are infinite dimensional but still have a Cartan structure, are generated by the procedure outlined in this section. In the next section an example is worked out which reveals this structure more clearly.

## 3. The algebra $\mathrm{A}_{l=2}^{\mathrm{SO}(3)}$

Let K = SO(3) and let $V$ be the five-dimensional representation space of the $l = 2$ representation of SO(3). Then a basis in $V$ can be written as $|2, m\rangle$, $m = -2, \ldots, 2$. $\mathscr{S}(V)$ is isomorphic to $HL_5^2$ and $z_m$ corresponds to $|2, m\rangle$.

The action of the Lie algebra of SO(3) on $HL_5^2$ is given by (see III, equation (5))

$$\begin{aligned}(L_3 f)(z) &= \sum_{m=-2}^{+2} m z_m D_m f(z)\\ (L_\pm f)(z) &= \sum_{m=-2}^{+2} c_m^\pm z_{m\pm 1} D_m f(z)\end{aligned} \tag{13}$$

where $c_m^{\pm}=[6-m(m\pm 1)]^{1/2}$. It is easily seen that $L_3$ and $L_{\pm}$ satisfy the usual SO(3) Lie algebra commutation relations.

To construct the algebra of operators commuting with the SO(3) action on $HL_5^2$, we need to find invariant polynomials $p^{(m)}(z)$. They are discussed in appendix 1 and are of the form

$$\begin{aligned} p^{(2)}(z)&=2z_2z_{-2}-2z_1z_{-1}+z_0^2 \\ p^{(3)}(z)&=12z_2z_0z_{-2}+6z_1z_0z_{-1}-2z_0^3-3\sqrt{6}z_2z_{-1}^2-3\sqrt{6}z_1^2z_{-2}. \end{aligned} \tag{14}$$

There are, of course, fourth- and higher-order polynomials, but they are all functions of $p^{(2)}$ and $p^{(3)}$.

The raising operators are now defined as

$$\begin{aligned} (X^{+2}f)(z)&=p^{(2)}(z)f(z) \\ (X^{+3}f)(z)&=p^{(3)}(z)f(z) \end{aligned} \tag{15}$$

along with the associated lowering operators $X^{-2}, X^{-3}$. A new raising operator $X^{+1}$ can be defined by

$$[X^{-2}, X^{+3}]\equiv 6X^{+1}. \tag{16}$$

$X^{+1}$ is no longer a multiplication operator, as are $X^{+2}$ and $X^{+3}$. It is of the form $z^2D$; its actual form is given in appendix 2, equation (A2.1). The factor 6 in equation (16) is chosen for convenience. The adjoint of $X^{+1}$ gives a lowering operator $X^{-1}$ of the form $zD^2$.

These three raising and lowering operators can be commuted among themselves to give the following commutation relations:

$$[X^{+2}, X^{+3}]=0 \tag{17a}$$

$$[X^{-2}, X^{+3}]=6X^{+1} \qquad [D^2, z^3]\sim z^2D \tag{17b}$$

$$[X^{+1}, X^{+2}]=2X^{+3} \qquad [z^2D, z^2]\sim z^3 \tag{17c}$$

$$[X^{-1}, X^{+2}]=4X^{+1} \qquad [zD^2, z^2]\sim z^2D \tag{17d}$$

$$[X^{+1}, X^{+3}]=12(X^{+2})^2 \qquad [z^2D, z^3]\sim z^4 \tag{17e}$$

$$[X^{-1}, X^{+3}]=84X^{+2}+24X^{+2}\hat{n} \qquad [z^2D, z^3]\sim z^2+z^3D. \tag{17f}$$

What distinguishes these commutators from those of a finite-dimensional Lie algebra are the quadratic terms on the right-hand side of equations (17*e*) and (17*f*). If $(X^{+2})^2$ were defined as a new element, and the commutator of this element with $X^{+m}$ were computed, higher-order elements would result, leading to an infinite-dimensional Lie algebra. However, by leaving the commutators in the form given in equation (17), it can be shown that they, along with other commutators, close and satisfy Jacobi identities.

The number operator $\hat{n}$ appears in equation (17*f*). It also appears in the commutator of $X^{\pm 2}$:

$$[X^{-2}, X^{+2}]=10+4\hat{n} \tag{18}$$

which along with

$$[\hat{n}, X^{\pm 2}]=\pm 2X^{\pm 2} \tag{19}$$

gives the algebra of $\mathrm{SL}(2, \mathbb{R})$, indicating that the $\mathrm{SL}(2, \mathbb{R})$ Lie algebra is a finite-dimensional subalgebra of $\mathrm{A}_2^{\mathrm{SO}(3)}$. Other commutators of $\hat{n}$ are

$$\begin{aligned} &[\hat{n}, X^{\pm 1}] = \pm X^{\pm 1} \\ &[\hat{n}, X^{\pm 3}] = \pm 3X^{\pm 3} \end{aligned} \tag{20}$$

justifying the notation that $X^{+m}$ (or $X^{-m}$) raises (lowers) the eigenvalue $n$ of $\hat{n}$ to $n+m$ $(n-m)$.

The most interesting commutators are $[X^{-1}, X^{+1}]$ and $[X^{-3}, X^{+3}]$, for they reveal the existence of a new Hermitian operator $X^0$, as discussed in § 2:

$$\begin{aligned} &[X^{-1}, X^{+1}] = 12\hat{n} + 16(\hat{n})^2 + 20X^{+2}X^{-2} - 6X^0 \\ &[X^{-3}, X^{+3}] = 420 + 252\hat{n} - 36X^{+2}X^{-2} + 18X^0. \end{aligned} \tag{21}$$

$X^0$ is a complicated operator, with 23 terms; as can be seen from the above commutators, it is of the form $z^2D^2$. The actual coefficients are given in appendix 2.

The last commutation relations of $\mathrm{A}_2^{\mathrm{SO}(3)}$ all involve $X^0$:

$$[X^0, \hat{n}] = 0 \tag{22}$$

$$[X^0, X^{+1}] = 6X^{+1} + 8X^{+1}\hat{n} \tag{23a}$$

$$[X^0, X^{+2}] = 20X^{+2} + 16X^{+2}\hat{n} \tag{23b}$$

$$[X^0, X^{+3}] = 42X^{+3} + 24X^{+3}\hat{n}. \tag{23c}$$

As discussed in § 2, $X^0$ commutes with $\hat{n}$, so these two operators generate a Cartan subalgebra. The remaining commutators, equation (23), show how the $X^{+m}$ act as raising operators for the eigenvalues of $X^0$.

Finally, there is a quadratic Casimir operator for $\mathrm{A}_2^{\mathrm{SO}(3)}$ of the form

$$C_{(2)}^{\mathrm{A}} = 4(\hat{n})^2 + 2\hat{n} - X^0. \tag{24}$$

We have not been able to find higher-order Casimir operators, or show that higher-order Casimir operators do not exist. The Casimir operator for SO(3) is $C_{(2)}^{\mathrm{SO}(3)} = L_-L_+ + L_3^2 + L_3$; after much tedious checking, it can be shown that $C_{(2)}^{\mathrm{A}} = C_{(2)}^{\mathrm{SO}(3)}$, generalising the result of Howe (1985) that dual pairs have Casimir operators with common eigenvalues.

Equations (17)-(23) give the commutation relations for the elements $X^{\pm m}$, $\hat{n}$, and $X^0$ of $\mathrm{A}_2^{\mathrm{SO}(3)}$. Since all of these elements commute with the SO(3) Lie algebra elements, equation (13), they can be used to distinguish between equivalent representations of SO(3) in $HL_5^2$. The decomposition of $n$-fold symmetric tensor products of $l=2$ representations of SO(3) is quite complicated. In the table below, this decomposition is given for small values of $n$:

| | $L=0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\mathrm{GL}(5, \mathscr{C})$ Rep. | Dim. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=0$ | × | | | | | | | | | | | (00000) | 1 |
| 1 | | | × | | | | | | | | | (10000) | 5 |
| 2 | × | | × | | × | | | | | | | (20000) | 15 |
| 3 | × | | × | × | × | | × | | | | | (30000) | 35 |
| 4 | × | | ×× | | ×× | × | × | | × | | | (40000) | 70 |
| 5 | × | | ×× | × | ×× | × | ×× | × | × | | × | (50000) | 126 |
| ⋮ | | | | | | | | | | | | ⋮ | ⋮ |

(25)

We note that for a given value of $n$, the maximum $L$ value is $2n$, as expected. However, for values of $L$ less than $2n$, the pattern is not clear. Multiplicity begins with $n=4$;

there are two $L=2$ and two $L=4$ states. Note also that there are gaps in some towers; for example, for $L=0$, there is no state between $n=0$ and $n=2$, and similarly there is no state for $L=3$ between $n=3$ and $n=5$, meaning that the raising operator $X^{+1}$ on such states must give 0.

To conclude this section we construct several examples of polynomials representing states of definite $L$ with multiplicity 1 in a number $n$ subspace. States with multiplicity will be treated in the next section. Consider, for example, the state $|L=3, L_3=3, n=3\rangle$ which, according to (25), has multiplicity 1. The polynomial corresponding to this state can be written as

$$p_{|3,3;3\rangle}(z) = \sum_{m_1 m_2 m_3} C^{32\ 2\ 2}_{3 m_1 m_2 m_3} z_{m_1} z_{m_2} z_{m_3}$$
$$= C_1 z_2 z_1 z_0 + C_2 z_1^3 + C_3 z_2^2 z_{-1}.$$

Here $C^{32\ 2\ 2}_{3 m_1 m_2 m_3}$ are Clebsch–Gordan coefficients which couple $2\otimes 2\otimes 2|_{\text{sym}}$ to $L=3$, $L_3=3$ states and the $C_i$ are the coefficients of the only polynomials that can contribute in the sum.

Applying $L_+$, equation (13$b$), to $p_{|3,3;3\rangle}(z)$ should give zero since $L_3=3$; this results in

$$p_{|3,3;3\rangle}(z) = 2z_2^2 z_{-1} - \sqrt{6} z_2 z_1 z_0 + z_1^3 \tag{26}$$

whose norm, calculated from the differentiation inner product defined in Klink and Ton-That (1979), is given by

$$(p_{|3,3;3\rangle}, p_{|3,3;3\rangle}) = 4\times 2! + 6 + 3! = 20$$
$$\hat{p}_{|3,3;3\rangle}(z) = \frac{1}{\sqrt{20}}(2z_2^2 z_{-1} - \sqrt{6} z_2 z_1 z_0 + z_1^3).$$

From this lowest element of the tower of $L=3$ representations, other states can be obtained by using the raising operators. For example

$$p_{|3,3;4\rangle}(z) = X^{+1} p_{|3,3;3\rangle} = 0$$
$$p_{|3,3;5\rangle}(z) = X^{+2} p_{|3,3;3\rangle}$$
$$= (2z_2 z_{-2} - 2z_1 z_{-1} + z_0^2)(2z_2^2 z_{-1} - \sqrt{6} z_2 z_1 z_0 + z_1^3). \tag{27}$$

Thus, once all the irreducible representations of $A_2^{SO(3)}$ have been given as lowest weights, all other states can be uniquely obtained by repeatedly using the raising operators.

## 4. The structure of irreducible representations of $A_2^{SO(3)}$

Let $|(\chi)n, \lambda\rangle$ be an element in the irreducible representation space of $A_2^{SO(3)}$ labelled by $(\chi)$. $n$ and $\lambda$ are eigenvalues of $\hat{n}$ and $X^0$, respectively; here we have assumed that there are no other independent operators in $A_2^{SO(3)}$ that commute with $\hat{n}$ and $X^0$. Since the Casimir operator $C^A$, equation (24), depends only on $\hat{n}$ and $X^0$, and is equal to $C^{SO(3)}$, we can write

$$4n^2 + 2n - \lambda = L(L+1) \tag{28}$$

and eliminate $\lambda$ as a state variable. Then $\chi$ is $L$ and a state can be written as $|L, n\rangle$. This will not yet uniquely label a state because of multiplicity.

Before dealing with this multiplicity, we consider the action of the operators $X^{\pm m}$ on $|L, n\rangle$. From equations (19) and (20) it follows that

$$\begin{aligned} \hat{n}X^{\pm m}|Ln\rangle &= (n \pm m)X^{\pm m}|Ln\rangle \\ X^{\pm m}|L, n\rangle &= c_n^{\pm m}(L)|L, n \pm m\rangle \end{aligned} \tag{29}$$

where $c_n^{\pm m}(L)$ are coefficients to be determined.

In particular, if $n = n_{\min}$, the lowest value of $n$ allowed in the irreducible representation $L$,

$$X^{-m}|L, n_{\min}\rangle = 0 \qquad m = 1, 2, 3 \tag{30}$$

and this defines the lowest weights of the irreducible representation $L$. As shown in appendix 3, for $L$ even, $n_{\min} = L/2$, and for $L$ odd, $n_{\min} = \frac{1}{2}(L+3)$.

To determine the coefficients $c_n^{\pm m}(L)$, we first note that $\langle L, n+m|X^{+m}|L, n\rangle = c_n^{+m}(L) = c_{n+m}^{-m}(L)^*$, where use has been made of the adjoint of $X^{+m}$ acting on $\langle L, n+m|$. Thus, as with finite-dimensional Lie algebras, the coefficients for the lowering operators are related to those of the raising operators.

Consider now commutators of the form $[X^{-m}, X^{+m}]$. The simplest of these is $[X^{-2}, X^{+2}]$, which, when applied to $|L, n\rangle$, gives

$$|c_n^{+2}(L)|^2 - |c_n^{-2}(L)|^2 = 10 + 4n. \tag{31}$$

If $n = n_{\min}$, then $c_{n_{\min}}^{-2}(L) = 0$ (from equation (30)) and

$$|c_{n_{\min}}^{+2}(L)|^2 = 10 + 4n_{\min}. \tag{32}$$

Equations (31) and (32), along with $c_n^{-2}(L)^* = c_{n-2}^{+2}(L)$ define $|c_n^{+2}(L)|^2$ recursively; adding everything up gives

$$|c_{n_{\min}+2k}^{+2}(L)|^2 = 10(k+1) + 4n_{\min}(k+1) + 4k(k+1) \qquad k = 0, 1, 2, \ldots. \tag{33}$$

Equation (33) gives $|c_n^{+2}(L)|^2$ only for $n = n_{\min} + 2k$. To get these coefficients for states with $n = n_{\min} + 2k + 1$, it is necessary to make use of the commutator $[X^{-1}, X^{+1}]$, equation (21), applied to a state $|L, n\rangle$:

$$\begin{aligned} |c_n^{+1}(L)|^2 - |c_n^{-1}(L)|^2 &= 12n + 16n^2 + 20|c_n^{-2}(L)|^2 - 6\lambda \\ &= 6L(L+1) + 20|c_n^{-2}(L)|^2 - 8n^2. \end{aligned} \tag{34}$$

For $n = n_{\min}$, this becomes

$$|c_{n_{\min}}^{+1}(L)|^2 = 6L(L+1) - 8n_{\min}^2 \tag{35}$$

and in particular $X^{+1}|L, n_{\min}\rangle = c_{n_{\min}}^{+1}(L)|L, n_{\min}+1\rangle$. Using equation (31) gives

$$|c_{n_{\min}+1}^{+2}(L)|^2 = 10 + 4(n_{\min}+1). \tag{36}$$

Here use has been made of the commutation relation $[X^{-2}, X^{+1}] = 4X^{-1}$ applied to $|L, n_{\min}\rangle$:

$$\begin{aligned} &X^{-2}X^{+1}|L, n_{\min}\rangle - X^{+1}X^{-2}|L, n_{\min}\rangle = 4X^{-1}|L, n_{\min}\rangle \\ &c_{n_{\min}}^{+1}(L)X^{-2}|L, n_{\min}+1\rangle - 0 = 0 \end{aligned}$$

from which it follows that $|c_{n_{\min}+1}^{-2}(L)|^2 = 0$.

Equations (31) and (36) can now be combined to give $|c_n^{+2}(L)|^2$ for $n = n_{\min} + 2k + 1$:

$$|c_{n_{\min}+2k+1}^{+2}(L)|^2 = 10(k+1) + 4n_{\min}(k+1) + 4(k+1)^2 \qquad k = 0, 1, 2, \ldots. \tag{37}$$

Equation (35) can also be used to show that there are no $L=1$ states. If $L=1$, then $|c^{+1}_{n_{\min}}(1)|^2=12-8n^2_{\min}$, and $n_{\min}=\frac{1}{2}(L+3)=2$ gives $|c^{+1}_2(1)|^2=-20$, which is impossible.

Note that for $L=0$ and $L=3$, $|c^{+1}_{n_{\min}}(L)|^2=6L(L+1)-8n^2_{\min}=0$ (i.e. for $L=0$, $n_{\min}=0$, and for $L=3$, $n_{\min}=3$). This means that the expression for $|c^{+2}_{n_{\min}+2k+1}(L)|^2$ is not valid for $L=0$, 3, since it was assumed in equation (36) that $c^{+1}_{n_{\min}}(L)\neq 0$. To modify $|c^{+2}_{n_{\min}+2k+1}(L)|^2$ for those cases where $c^{+1}_{n_{\min}}(L)=0$, we use the commutator $[X^{+3}, X^{-2}]=-6X^{+1}$:

$$X^{+3}X^{-2}|L, n_{\min}\rangle - X^{-2}X^{+3}|L, n_{\min}\rangle = -6X^{+1}|L, n_{\min}\rangle$$

$$0-c^{+3}_{n_{\min}}(L)X^{-2}|L, n_{\min}+3\rangle=0$$

so $c^{-2}_{n_{\min}+3}(L)=0$. Similarly, $[X^{-1}, X^{+2}]=4X^{+1}$ gives

$$X^{-1}X^{+2}|L, n_{\min}\rangle - X^{+2}X^{-1}|L, n_{\min}\rangle = 4X^{+1}|L, n_{\min}\rangle$$

$$c^{+2}_{n_{\min}}(L)X^{-1}|L, n_{\min}+2\rangle-0=0$$

which means $c^{-1}_{n_{\min}+2}(L)=0$.

Finally, to compute $|c^{+2}_{n_{\min}+2k+3}(L)|^2$ when $c^{+1}_{n_{\min}}(L)=0$, use is made of the commutator $[X^{-3}, X^{+3}]$ applied to $|L, n\rangle$:

$$\begin{aligned}|c^{+3}_n(L)|^2-|c^{-3}_n(L)|^2 &= 420+252n-36|c^{-2}_n(L)|^2+18\lambda\\ &= 420-18L(L+1)+72n^2+288n-36|c^{-2}_n(L)|^2\end{aligned}$$

$$|c^{+3}_{n_{\min}}(L)|^2=420-18L(L+1)2n^2_{\min}+288n_{\min}. \tag{38}$$

From equation (31) and the fact that $|c^{-2}_{n_{\min}+3}(L)|^2=0$, we get

$$|c^{+2}_{n_{\min}+3}(L)|^2=10+4(n_{\min}+3). \tag{39}$$

Then the general coefficient for $n=n_{\min}+2k+3$ becomes

$$|c^{+2}_{n_{\min}+2k+3}(L)|^2=10(k+1)+4(n_{\min}+3)(k+1)+k(k+1) \qquad k=0,1,2,\ldots. \tag{40}$$

Thus far we have assumed that, for a given value of $L$, a state can be uniquely labelled by $n$. However, multiplicity occurs for most states, which means $n$ does not uniquely label a state. Just as with SU(3), where a state is not uniquely labelled by the third component of isospin and hypercharge, but needs an additional label that is chosen to be the total isospin, so too with $\mathrm{A}_2^{\mathrm{SO}(3)}$, an additional label can be chosen as the eigenvalue of the Casimir operator of the $\mathrm{SL}(2,\mathbb{R})$ subalgebra. $C^{\mathrm{SL}(2,\mathbb{R})}=(\hat{n})^2+3\hat{n}-X^{+2}X^{-2}$ commutes with $X^{\pm2}$, $\hat{n}$, $X^0$ and $C^A$, but not with $X^{\pm1}$ and $X^{\pm3}$. Since towers of states have been constructed by the repeated use of $X^{+2}$, resulting in states $|L, n_{\min}+2k\rangle$ and $|L, n_{\min}+2k+1\rangle$, these towers will all have a fixed eigenvalue of $C^{\mathrm{SL}(2,\mathbb{R})}=n^2_{\min}+3n_{\min}$.

Now states with different eigenvalues of the $\mathrm{SL}(2,\mathbb{R})$ Casimir operator are orthogonal. States of multiplicity 2 already occur for $n=n_{\min}+2$; we want to construct states with the same value of $n=n_{\min}+2$ to have different eigenvalues of $C^{\mathrm{SL}(2,\mathbb{R})}$. Consider $X^{+2}\|L, n_{\min}\rangle=c^{+2}_{\min}(L)|L, n_{\min}+2\rangle$. In general, the state $(X^{+1})^2|L, n_{\min}\rangle$ will not be proportional to $|L, n_{\min}+2\rangle$. So define new states $|L, N; n\rangle$, where $N$ is the lowest value of $n$ in the $\mathrm{SL}(2,\mathbb{R})$ tower, i.e. $X^{-2}|L, N; N\rangle=0$, which means $C^{\mathrm{SL}(2,\mathbb{R})}|L, N; n\rangle=(N^2+3N)|L, N, n\rangle$.

A linear combination of $X^{+2}|L, n_{\min}\rangle$ and $|L, N=n_{\min}+2; n_{\min}+2\rangle$ can then be written as the state $(X^{+1})^2|L, n_{\min}\rangle$:

$$(X^{+1})^2|L, n_{\min}\rangle = \alpha X^{+2}|L, n_{\min}\rangle + \beta|L, n_{\min}+2, n_m+2\rangle \tag{41}$$

where $\alpha$ and $\beta$ are constants to be determined. Applying $X^{-2}$ to both sides of equation (41) gives

$$X^{-2}(X^{+1})^2|L, n_{\min}\rangle = \alpha X^{-2}X^{+2}|L, n_{\min}\rangle$$
$$[(X^{+1})^2X^{-2} + 4(X^{+1}X^{-1} + X^{-1}X^{+1})]|L, n_{\min}\rangle = \alpha(10+4n_{\min})|L, n_{\min}\rangle$$

where use has been made of the fact that $X^{-2}|L, n_{\min}+2, n_{\min}+2\rangle = 0$ and $[X^{-2}, (X^{+1})^2] = 4(X^{+1}X^{-1} + X^{-1}X^{+1})$. Using the commutator $[X^{-1}, X^{+1}]$, equation (21) finally gives

$$\alpha = \frac{4[6L(L+1) - 8n_{\min}^2]}{10+4n_{\min}}. \tag{42}$$

Knowledge of $\alpha$ can be used to calculate $|\beta|^2$; using equation (41)

$$|\beta|^2 = \langle L, n_{\min}|((X^{-1})^2 - \alpha X^{-2})((X^{+1})^2 - \alpha X^{+2})|L, n_{\min}\rangle$$

which, after some tedious algebra, gives

$$|\beta|^2 = [6L(L+1) - 8n_{\min}^2]^2\left(\frac{2+4n_{\min}}{5+2n_{\min}} - \frac{4(2n_{\min}+1)}{3L(L+1) - 4n_{\min}^2}\right). \tag{43}$$

This result can be used to find when there is no multiplicity for $n = n_{\min}+2$ states, for then $|\beta| = 0$. For example, if $L=2$ and $n_{\min}=1$, $\beta = 0$, indicating that the $L=2$, $n=3$ state has multiplicity 1. Similarly, if $L=5$ and $n_{\min} = \frac{1}{2}(L+3) = 4$, then $\beta = 0$, so the $L=5$, $n=6$ state also has multiplicity 1. Conversely, when $\beta \neq 0$, the state $n = n_{\min}+2$ will always have multiplicity 2 and the multiplicity is broken by the eigenvalues of the $SL(2, \mathbb{R})$ Casimir operator.

The states $|L, n_{\min}+2, n = n_{\min}+2+2k\rangle$, $k = 0, 1, 2, \ldots$, can be written as $(X^{+2})^k|L, n_{\min}+2, n_{\min}+2\rangle$, and the coefficients $c^{+2}_{n_{\min}+2+2k}(L)$ determined from the recursion relation, equation (11), with $c^{-2}_{n_{\min}+2}(L) = 0$. The result is

$$|c^{+2}_{n_{\min}+2+2k}(L)|^2 = 10(k+1) + 4(n_{\min}+2)(k+1) + k(k+1) \qquad k = 0, 1, 2, \ldots. \tag{44}$$

If one tries to go to $n = n_{\min}+3$ and higher states, the multiplicity pattern becomes very complicated; we have not been able to find a general expression for the multiplicity of a state with given $L$ and $n$ and it is not clear whether the eigenvalues of $C^{SL(2,\mathbb{R})}$ alone will break all the multiplicity.

## 5. Conclusion

The notion of dual pairs of groups can be generalised to include a much larger class than discussed by Moshinsky and Quesne (1970, 1971) and Howe (1985). In this paper we have shown how to construct a dual algebra of operators that commutes with a compact group action on a Fock space generated by *any* finite-dimensional representation space.

The main idea is to replace the Fock space by an isomorphic holomorphic Hilbert space and look for polynomials that are invariant under the compact group action. It is always possible to find a finite set of such polynomials that generate all the remaining invariant polynomials. This finite set can be used to define raising and lowering operators, whose commutators then generate a Lie algebra of operators that commutes with the compact group action. If the independent polynomial invariants are of first or second degree, the algebra will be finite dimensional and we recover the results of Moshinsky and Quesne (1970, 1971) and Howe (1985).

However, if any of the independent polynomials are of degree greater than two, the resulting algebra will always be infinite dimensional. Such was the case for the algebra generated by the $l=2$ representation of SO(3) discussed in § 3. It seems likely that there is a very large class of infinite-dimensional Lie algebras generated by higher-degree invariant polynomials; what their relationship is—if any—to Virasoro or Kac–Moody algebras is not at all clear. It is also not clear under what circumstances these algebras can be exponentiated to give groups that are dual in the sense of Howe (1985).

Once the structure of the algebra is known, its representations can be used to label all the equivalent representations of the compact group on the Fock space. The irreducible representations of the algebra are given by lowest weights and all other states can be reached by raising operators. In particular, if the polynomial realisation of a lowest weight is known, the polynomial realisation of any higher weight can be computed by using the concrete realisation of the raising operators as multiplication or differentiation operators. This means that the Clebsch–Gordan coefficients of any symmetric $n$-fold tensor product of any irreducible representation of a compact group can quite easily be computed. A simple example is given in appendix 1.

Using the differentiation inner product, it is also quite easy to compute the matrix elements of algebra elements. This will be of interest in the succeeding paper, where the compact group is $\mathrm{SU}(3)_{\mathrm{flavour}}$, and the representation space generating the Fock space comes from the eight pseudoscalar mesons. Such matrix elements are also of interest in other applications such as the interacting boson model in nuclear physics.

In this paper we have analysed in some detail the algebra of $\mathrm{A}_2^{\mathrm{SO}(3)}$ and have shown that the irreducible representations of $\mathrm{A}_2^{\mathrm{SO}(3)}$ can be specified by lowest weights $n_{\min}$ relative to the lowering operators $X^{-1}, X^{-2}, X^{-3}$. Such lowest-weight states are unique and are labelled by the angular momentum $L$ of SO(3), with $n_{\min}=L/2$ for $L$ even and $n_{\min}=\frac{1}{2}(L+3)$ for $L$ odd, the only exception being $L=1$, which does not occur as an irreducible representation.

From the lowest weight, all other states can be reached by applying the raising operators. But these states are not uniquely specified by $L$ and $n \geqslant n_{\min}$. One way to deal with the multiplicity is to introduce the eigenvalue of the Casimir operator of the $\mathrm{SL}(2, \mathbb{R})$ subalgebra, or what is equivalent, to introduce as an additional state label $N$, the lowest weight in the $\mathrm{SL}(2, \mathbb{R})$ discrete series of representations. Then a general state can be written as $|L, N, n\rangle$ where $n \geqslant N \geqslant n_{\min}$ and $X^{-2}|L, N, n=N\rangle=0$, with the eigenvalue of the $\mathrm{SL}(2, \mathbb{R})$ Casimir operator given by $N^2+3N$.

Towers of states can be obtained from low-lying weights by applying the raising operator $X^{+2}$ $k$ times. The simplest such tower is $|L, N=n_{\min}, n=n_{\min}+2k\rangle$, $k=0,1,2,\ldots$, with

$$(X^{+2})^k|L, n_{\min}\rangle = C^{+2}_{n_{\min}+2k}(L)|L, n_{\min}, n_{\min}+2k\rangle.$$

Similarly, another tower of states has the form $|L, N=n_{\min}, n=n_{\min}+2k+1\rangle$ which are obtained from

$$(X^{+2})^k(X^{+1})|L, n_{\min}\rangle = c^{+2}_{n_{\min}+2k+1}(L)c^{+1}_{n_{\min}}(L)|L, n_{\min}+1, n_{\min}+2k+1\rangle$$

where $c^{+1}_{n_{\min}}(L)=[6L(L+1)-8n^2_{\min}]^{1/2}$ (equation (35)).

If $c^{+1}_{n_{\min}}(L)=0$, then the states in the tower $n=n_{\min}+2k+3$ are given by

$$(X^{+2})^k X^{+3}|L, n_{\min}\rangle = c^{+2}_{n_{\min}+2k+3}(L)c^{+3}_{n_{\min}}(L)|L, n_{\min}+3, n_{\min}+2k+3\rangle$$

with

$$c^{+3}_{n_{\min}}(L)=[420-18L(L+1)+72n^2_{\min}+288n_{\min}]^{1/2}$$

(equation (38)).

Multiplicity already appears for $n=n_{\min}+2$; states of the form $|L, N=n_{\min}+2, n=2k+2\rangle$ will be orthogonal to $|L, N=n_{\min}, n=2k+2\rangle$ because the eigenvalues of the $\mathrm{SL}(2,\mathbb{R})$ Casimir operator are different. Since by definition $X^{-2}|L, N=n_{\min}+2, n=N\rangle=0$, a new tower of orthogonal states is generated by

$$(X^{+2})^k|L, n_{\min}+2, n=n_{\min}+2\rangle = c^{+2}_{n_{\min}+2k+2}(L)|L, n_{\min}+2, n_{\min}+2k+2\rangle$$

where the state $|L, n_{\min}+2, n_{\min}+2\rangle$ comes from the space of $X^{+2}|L, n_{\min}, n_{\min}\rangle$ and $(X^{+1})^2|L, n_{\min}, n_{\min}\rangle$ (see equation (41)ff). Not all $n=n_{\min}+2$ states have multiplicity; there is no multiplicity if $\beta=0$ (equation (42)).

All of the coefficients $c^{+2}_n(L)$ in the preceding equations have the form $c^{+2}_{n_{\min}+2k+k_0}(L)$. The calculations for $c^{+2}_n(L)$ given in § 4, namely equations (33), (37), (40) and (44), with $k_0=0, 1, 3$, and 2, respectively, can all be combined into the general expression

$$c^{+2}_{n_{\min}+2k+k_0}(L)=[10(k+1)+4(n_{\min}+k_0+k)(k+1)]^{1/2}.$$

The justification for assuming $c^{+m}_n(L)$ is real comes from the concrete realisation of the states $|L, N, n\rangle$ as polynomials. When the states are realised as polynomials, the raising operators become multiplication and differentiation operators, and when applied to the lowest-weight polynomials give the states $|L, N, n\rangle$. The $c^{+m}_n(L)$ coefficients are computed from the norms of the polynomials and can also be chosen to be real.

It is possible to continue the procedure whereby at level $n\geqslant n_{\min}+3$, the multiplicity of states is computed and new towers of states are generated with $(X^{+2})^k$. However, the analysis already is very complicated for $n=n_{\min}+3$, reflecting the fact that the general multiplicity structure of $\mathrm{A}_2^{\mathrm{SO}(3)}$ is very complicated. It remains to carry out a general multiplicity analysis and see if all the multiplicity can be dealt with using the eigenvalues of the $\mathrm{SL}(2,\mathbb{R})$ Casimir operator. In fact, it is not even clear whether all of the irreducible representations of $\mathrm{A}_2^{\mathrm{SO}(3)}$ are given by $n_{\min}=L/2$, $L$ even and $n_{\min}=\frac{1}{2}(L+3)$, $L$ odd. It is, of course, always possible to determine the multiplicity structure for a given $n$ by decomposing the symmetric $n$-fold tensor product of $l=2$ representations of SO(3) along the lines given in the appendix. However, while this is useful for practical calculations, it does not reveal very much about the general multiplicity structure.

The complicated multiplicity structure makes the analysis of the irreducible representations of the infinite-dimensional Lie algebras $\mathrm{A}_V^{\mathrm{K}}$ more complicated than their finite-dimensional counterparts such as $\mathrm{SL}(2,\mathbb{R})$. However, in other respects, as seen in this paper, the towers of states are no more complicated for infinite-dimensional Lie algebras than for finite-dimensional ones.

## Acknowledgments

I wish to thank Professors P Jorgensen and T Ton-That for many helpful conversations, and Mr R Wills for his computer skills, used to compute many of the complicated commutators of $\mathrm{A}_2^{\mathrm{SO}(3)}$. This work was supported in part by the DoE.

## Appendix 1. Clebsch–Gordan coefficients and invariant polynomials

It is often necessary to transform between symmetric $n$-fold tensor product states and states which transform as an irreducible representation of K. In particular, to compute the invariant polynomials which generate the algebra $A_V^K$, it is necessary to know the Clebsch–Gordan coefficients for the identity representation of K in the decomposition of $n$-fold tensor products.

Consider first individual particle states in the $n$-fold tensor product. They can be written as $|\{n_j\}\rangle$, $\Sigma_{j=1}^N n_j = n$. The unnormalised polynomial corresponding to this state is $\Pi_{j=1}^N z_j^{n_j}$. Its norm is given by $(n_1! \ldots n_N!)^{1/2}$, so that

$$|\{n_j\}\rangle = \prod_{j=1}^{N} z_j^{n_j}(n_1! \ldots n_N!)^{-1/2}.$$

As discussed in § 2, a state in the $n$-particle subspace transforming as the irreducible representation $\chi$ of K, with basis labels $i$ can be written as $|\chi i n \eta\rangle$, where $\eta$ is a multiplicity label. Related to this state is the unnormalised polynomial $p_{|\chi i n \eta\rangle}$. Then

$$\hat{p}_{|\chi i n \eta\rangle}(z) = \frac{p_{|\chi i n \eta\rangle}(z)}{\|p_{|\chi i n \eta}\|}.$$

$p_{|\chi i n \eta\rangle}(z)$ can be obtained from $p_{|\chi i n_{\min} \eta\rangle}(z)$ by using raising operators; $p_{|\chi i n_{\min} \eta\rangle}(z)$ is generally easy to compute by using raising operators in the Lie algebra of K, as discussed in the example at the end of § 3, equations (26) and (27).

Then

$$\langle\{n_j\}|\chi i n \eta\rangle = (n_1! \ldots n_N!)^{-1/2} \frac{1}{\|p_{|\chi i n \eta}\|} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_N}}{\partial z_N^{n_N}} p_{|\chi i n \eta\rangle}|_{z=0}.$$

For example, the polynomial given in equation (26), $p_{|3,3;3\rangle}(z)$ has norm $\|p_{|3,3;3\rangle}\| = \sqrt{20}$, and a possible three-particle state is $n_2 = 2$, $n_{-1} = 1$, all other $n_j = 0$, which can be written as $\hat{p}_{|2,0,0,1,0\rangle}(z) = (1/\sqrt{2}) z_2^2 z_{-1}$. Then

$$\left\langle \begin{matrix} n_2 n_1 n_0 n_{-1} n_{-2} L L_3 n \\ 2, 0, 0, 1, 0 | 3, 3; 3 \end{matrix} \right\rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{20}} \frac{\partial^2}{\partial z_2^2} \frac{\partial}{\partial z_{-1}} p_{|3,3;3\rangle}|_{z=0}$$

$$= \frac{1}{\sqrt{40}} 4.$$

The invariant polynomials $p^{(m)}(z)$ can always be written as Clebsch–Gordan coefficients which couple an $m$-fold tensor product to give the identity representation of K. The form is given in equation (5). However, now the polynomials are not known. They can be obtained by writing down the most general linear combination of $m$ products of $z_i$ which give $L_3 = 0$, and then using the raising operator $L_+$, equation (13), to determine the actual linear combinations. When the polynomials $p^{(2)}(z)$ and $p^{(3)}(z)$, equation (14), are normalised, the coefficients will then be the Clebsch–Gordan coefficients, equation (5):

$$\|p^{(2)}\|^2 = 4 + 4 + 2! = 10$$

$$\|p^{(3)}\|^2 = 12^2 + 6^2 + 4 \times 3! + 9 \times 6 \times 2! + 9 \times 6 \times 2! = 420$$

$$\hat{p}^{(2)}(z)=\frac{2}{\sqrt{10}}z_2z_{-2}-\frac{2}{\sqrt{10}}z_1z_{-1}+\frac{1}{\sqrt{10}}z_0^2$$

$$\hat{p}^{(3)}(z)=\left(\frac{12}{35}\right)^{1/2}z_2z_0z_{-2}+\left(\frac{3}{35}\right)^{1/2}z_1z_0z_{-1}-\frac{1}{\sqrt{105}}z_0^3-\frac{3}{\sqrt{70}}z_2z_{-1}^2-\frac{3}{\sqrt{70}}z_1^2z_{-2}.$$

So, for example,

$$C^{0\,2\,2\,2}_{0\,2\,-1\,-1}=-3/\sqrt{70}.$$

## Appendix 2. Concrete realisations of the (basis) operators of $A_2^{SO(3)}$ and Jacobi identities

The operators $X^{+2}$ and $X^{+3}$ were defined in equations (14) and (15) as multiplication operators. However, $X^{+1}$ is not a multiplication operator. It is defined through the commutator $[X^{-2}, X^{+3}]\equiv 6X^{+1}$, which gives

$$\begin{aligned}X^{+1}=&(4z_2z_0-\sqrt{6}z_1^2)D_2+(2\sqrt{6}z_2z_{-1}-2z_1z_0)D_1+(4z_2z_{-2}+2z_1z_{-1}-2z_0^2)D_0\\&+(2\sqrt{6}z_1z_{-2}-2z_0z_{-1})D_{-1}+(4z_0z_{-2}-\sqrt{6}z_{-1}^2)D_{-2}.\end{aligned}\tag{A2.1}$$

By far the most complicated operator is $X^0$, defined via the commutators $[X^{-1}, X^{+1}]$ and $[X^{-3}, X^{+3}]$, in equation (21). The commutator $[X^{-3}, X^{+3}]\sim[D^3, z^3]$, so that the most general form of this commutator is

$$[X^{-3}, X^{+3}]=a_3+b_3\hat{n}+c_3(\hat{n})^2+d_3X^{+2}X^{-2}+e_3X^0.\tag{A2.2}$$

The Jacobi identity

$$[X^{+1},[X^{-3},X^{+3}]]+[X^{-3},[X^{+3},X^{+1}]]+[X^{+3},[X^{+1},X^{-3}]]=0$$

forces $d_3=-36$, as well as giving relations between the other constants.

By using the known expressions for $X^{+3}$ and $X^{-3}$, we have computed $[X^{-3}, X^{+3}]$ by using the symbolic manipulation program SMP on a VAX 11/780. The result is

$$[X^{-3}, X^{+3}]=420+252\hat{n}-36\sqrt{6}X_A^0+54X_B^0+36X_C^0\tag{A2.3}$$

where

$$\begin{aligned}X_A^0=&z_{-1}^2D_0D_{-2}+z_1^2D_2D_0+z_0z_{-2}D_{-1}^2+z_2z_0D_1^2+z_1z_{-2}D_0D_{-1}\\&+z_2z_{-1}D_1D_0+z_0z_{-1}D_1D_{-2}+z_1z_0D_2D_{-1}\\X_B^0=&z_1^2D_1^2+z_{-1}^2D_{-1}^2\\X_C^0=&z_0^2D_0^2-2z_0^2D_2D_{-2}-z_0^2D_1D_{-1}+4z_0z_{-2}D_0D_{-2}+6z_1z_{-2}D_1D_{-2}-2z_2z_{-2}D_0^2\\&+4z_2z_{-2}D_2D_{-2}+2z_2z_{-2}D_1D_{-1}+z_0z_{-1}D_0D_{-1}-z_1z_{-1}D_0^2\\&+2z_1z_{-1}D_2D_{-2}+z_1z_0D_1D_0+4z_2z_0D_2D_0.\end{aligned}\tag{A2.4}$$

Equating (A2.2) with (A2.3) gives $a_3=420$, $b_3=252$, $c_3=0$, and $-36X^{+2}X^{-2}+e_3X^0=-36\sqrt{6}X_A^0+54X_B^0+36X_C^0$, or

$$e_3X^0=18(-2\sqrt{6}X_A^0+3X_B^0+2X_C^0+2X^2X^{-2}).\tag{A2.5}$$

All of the terms in $X^{+2}X^{-2}$ appear in $X_C^0$. We choose $e_3=18$ for convenience, which then defines $X^0$.

Similarly, the commutator of $[X^{-1}, X^{+1}]$ can be written as

$$[X^{-1}, X^{+1}]=a_1+b_1\hat{n}+c_1(\hat{n})^2+d_1X^{+2}X^{-2}+e_1X^0$$

only now it is not necessary to explicitly work out the commutator, using equation (A2.1) and its adjoint. Rather, Jacobi identities can be used to give relations between the various terms in the commutators. For example, the Jacobi identity

$$[X^{-3},[X^{+1},X^{+2}]]+[X^{+1},[X^{+2},X^{-3}]]+[X^{+2},[X^{-3},X^{+1}]]=0$$

can be used to show that $a_1=0, b_3+3b_1=288, c_3+3c_1=48, d_3+3d_1=24$, and $e_3+3e_1=0$, resulting in equation (21).

To fix the constants in the commutators for $[X^0, X^m]$, equation (23), other Jacobi identities such as

$$X^0,[X^{+1},X^{+3}]]+[X^{+1},[X^{+3},X^0]]+[X^{+3},[X^0,X^{+1}]]=0$$

can be used. Finally, many of the remaining Jacobi identities were used to check the overall consistency of the commutation relations.

## Appendix 3. Relation between $L$ and lowest weight $n_{\min}$

If a symmetric $n$-fold tensor product of SO(3), $l=2$, representations is decomposed, the highest weight that occurs is $L=2n$, which for $L_3=2n$, is realised by the polynomial $z_2^n$. Applying the SO(3) lowering operator $L_-$ of equation (13) results in the following sequence of states:

$$\underset{L_3=2n}{z_2^n} \xrightarrow{L_-} \underset{L_3=2n-1}{z_2^{n-1}z_1} \xrightarrow{L_-} \underset{L_3=2n-2}{z_2^{n-2}z_1^2,\ z_2^{n-1}z_0} \xrightarrow{L_-} \underset{L_3=2n-3}{z_2^{n-3}z_1^3,\ z_2^{n-2}z_1z_0,\ z_2^{n-1}z_{-1}}$$

The $L_3=2n-1$ state is part of the $L=2n$ multiplet. However, for $L_3=2n-2$, one linear combination of the two possible states will be in $L=2n$, while the other will be in $L=2n-2$. But $L=2n-2$ already occurs as the highest weight in the $(n-1)$-fold tensor product decomposition. For $L_3=2n-3$, the possible $L$ values are $L=2n, 2n-2$ and $2n-3$, with $L=2n-3$ occurring for the first time. Turning this around, for $L$ even the lowest value of $n$ is $n_{\min}=L/2$, whereas for $L$ odd, $L=2n_{\min}-3$, or $n_{\min}=\frac{1}{2}(L+3)$.

As a further check on this result, it is straightforward to show that $X^{-m}$, $m=1,2,3$, realised as differential operators acting on the $L=2n-3$, $L_3=2n-3$ polynomial states annihilate these states, indicating they are lowest-weight states.

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